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A measure-theoretic Grothendieck inequality

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ABSTRACT

In this note we develop a notion of integration with respect to a bimeasure μ that allows integration of functions in the projective tensor product $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$, where ν_1 and ν_2 are Grothendieck measures for μ. This integral, which agrees with the standard notion of integration with respect to a bimeasure, allows us to integrate inner products and provides a generalization of the Grothendieck inequality to a measure-theoretic setting.

1. Introduction

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For f and g bounded functions on X and Y (respectively) taking values in H, define the scalar-valued function $\langle f, g \rangle : X \times Y \to \mathbb{R}$ by

$$\langle f, g \rangle(x, y) = \big\langle f(x), g(y) \big\rangle, \quad (x, y) \in X \times Y. \tag{1}$$

Denote by $F_2(\mathcal{A}, \mathcal{B})$ the space of all bimeasures on $\mathcal{A} \times \mathcal{B}$; that is, the space of set functions on $\mathcal{A} \times \mathcal{B}$ which are countably additive in each argument separately [2, Definition VI.3.2]. The norm on $F_2(\mathcal{A}, \mathcal{B})$ is denoted by $\| \cdot \|_{F_2}$.

In this paper, we show that $\langle f, g \rangle$ can be (unambiguously) integrated with respect to any $\mu \in F_2(\mathcal{A}, \mathcal{B})$, whenever f and g are bounded and have separable range (i.e., f and g are *strongly measurable*), and then prove a generalization of the Grothendieck inequality. In particular, we prove the following theorem.

Theorem 1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let H be a separable Hilbert space. Suppose $f : X \to H$ and $g : Y \to H$ are bounded measurable functions. For any $\mu \in F_2(\mathcal{A}, \mathcal{B})$,*

$$\int \langle f, g \rangle \, d\mu = \lim_{N \to \infty} \int \left(\sum_{j=1}^{N} f_j(x) g_j(y) \right) \mu(dx, dy), \tag{2}$$

where $(f_j)_{j=1}^{\infty}$ and $(g_j)_{j=1}^{\infty}$ are the coordinate functions of f and g (respectively) with respect to an orthonormal basis of H. Furthermore,

$$\left| \int \langle f, g \rangle \, d\mu \right| \leqslant K_G \|\mu\|_{F_2} \sup_{x \in X} \big\| f(x) \big\|_H \sup_{y \in Y} \big\| g(y) \big\|_H,$$

where K_G is the Grothendieck constant.

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If $X = Y = \mathbb{N}$, this inequality is the Grothendieck inequality as formulated by Lindenstrauss and Pełczyński [4, Theorem 2.1].

We show that the integral in (2) is independent of choice of basis, and indeed independent of pointwise representation (Theorem 9). Furthermore, this integral will agree with the standard integral in the event that $\langle f, g\rangle$ is in the projective tensor product $L^\infty(X)\hat{\otimes}L^\infty(Y)$ (Proposition 10).

Throughout this note, $\mathbb{R}$ is the scalar field, but the results can be adapted to $\mathbb{C}$. We will also take H to be a separable Hilbert space, and we note that the notions of strong measurability and weak measurability coincide in this case. Consequently, without ambiguity we refer to functions as measurable.

2. Motivation

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be arbitrary measurable spaces. We wish to show that the function $\langle f, g\rangle : X \times Y \to \mathbb{R}$ defined in (1) is integrable with respect to any $\mu \in F_2(\mathcal{A}, \mathcal{B})$. A natural space of functions integrable with respect to any bimeasure is the projective tensor product $V_2(X, Y) = L^\infty(X, \mathcal{A}) \hat{\otimes} L^\infty(Y, \mathcal{B})$ (see, for example, [2]). In general, however, the function $\langle f, g\rangle$ need not be in $V_2(X, Y)$. This is illustrated in the following argument, shown to the author by R.C. Blei.

Proposition 2. *Let $E = \{\lambda_j\}_{j\in\mathbb{N}}$ be a lacunary set in $\mathbb{Z}^+$. There exists a Hilbert space H, and bounded functions $f : E \to H$ and $g : E \to H$ such that the function*

$$\langle f, g\rangle(\lambda_m, \lambda_n) = \big\langle f(\lambda_m), g(\lambda_n)\big\rangle_H, \quad m, n \in \mathbb{N},$$

is not in $V_2(E, E)$.

Proof. In [5], Varopoulos showed the proper inclusion $V_2(E, E) \subset B(E \times E)$, where $B(E \times E)$ is the Banach algebra of transforms of measures on $\mathbb{T} \times \mathbb{T}$ restricted to $E \times E$.

Let ϕ be an element of $B(E \times E)$ that is not in $V_2(E, E)$. By definition of $B(E \times E)$, there exists a measure μ on $\mathbb{T} \times \mathbb{T}$ such that, for all $m, n \in \mathbb{N}$,

$$\phi(\lambda_m, \lambda_n) = \int\limits_{\mathbb{T}\times\mathbb{T}} e^{-i\lambda_m s}e^{-i\lambda_n t}\, \mu(ds, dt). \tag{3}$$

The measure μ defines a bounded, bilinear functional $\tilde{\mu}$ on $C(\mathbb{T}) \times C(\mathbb{T})$:

$$\tilde{\mu}(h, k) = \int\limits_{\mathbb{T}\times\mathbb{T}} h(s)k(t)\, \mu(ds, dt), \quad h, k \in C(\mathbb{T}).$$

Let $\nu_1(\cdot) = |\mu|(\cdot, \mathbb{T})$ and $\nu_2(\cdot) = |\mu|(\mathbb{T}, \cdot)$. By the Cauchy–Schwarz inequality,

$$\big|\tilde{\mu}(h, k)\big| \leqslant \|\mu\|_M \|h\|_{L^2(\nu_1)} \|k\|_{L^2(\nu_2)}, \quad h, k \in C(\mathbb{T}).$$

Therefore, μ extends to a bounded, bilinear functional on $L^2(\mathbb{T}, \nu_1) \times L^2(\mathbb{T}, \nu_2)$. There exists then a bounded linear map $T : L^2(\mathbb{T}, \nu_2) \to L^2(\mathbb{T}, \nu_1)$ such that

$$\tilde{\mu}(h, k) = \big\langle h, T(k)\big\rangle_{L^2(\mathbb{T},\nu_1)}, \quad h \in L^2(\mathbb{T}, \nu_1),\ k \in L^2(\mathbb{T}, \nu_2).$$

For $n \in \mathbb{N}$, let $e_n(t) = e^{-int}$, $t \in \mathbb{T}$. Define $f, g : E \to L^2(\mathbb{T}, \nu_1)$ by

$$f(\lambda_m) = e_{\lambda_m}, \qquad g(\lambda_n) = T(e_{\lambda_n}), \quad m, n \in \mathbb{N}.$$

It follows that

$$\phi(\lambda_m, \lambda_n) = \big\langle f(\lambda_m), g(\lambda_n)\big\rangle_{L^2(\mathbb{T},\nu_1)}, \quad m, n \in \mathbb{N}.$$

By assumption, $\phi \notin V_2(E, E)$, and hence the result. $\square$

3. The spaces $L^2(\nu_1)\hat{\otimes}L^2(\nu_2)$

In this section, we find the spaces we need for our generalization of the Grothendieck inequality. We begin by recalling the following theorem.

Theorem 3 *(Grothendieck factorization theorem). Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let $\mu \in F_2(\mathcal{A}, \mathcal{B})$. There exist probability measures ν_1 on X and ν_2 on Y such that*

$$\left|\int f \otimes g\, d\mu\right| \leqslant K_G \|\mu\|_{F_2} \|f\|_{L^2(\nu_1)} \|g\|_{L^2(\nu_2)}, \quad (f, g) \in L^\infty(X) \times L^\infty(Y). \tag{4}$$

This was first proved for X and Y compact Hausdorff spaces (for a detailed proof, see Theorem V.2 in [2]). The extension to general measurable spaces was obtained by K. Ylinen in [6] (Lemma 2.1) using the Yosida–Hewitt decomposition theorem for finitely additive measures (Theorem 1.23 in [7] or Theorem 10.2.1 in [1]).

We make the following observation: Suppose $\mu \in F_2(\mathcal{A}, \mathcal{B})$, and ν_1 and ν_2 are Grothendieck measures obtained from Theorem 3. Let $f \in L^2(\nu_1)$ and $g \in L^2(\nu_2)$. By density of simple functions, there exist functions $(f_n)_{n=1}^\infty$ in $L^\infty(X)$, and $(g_n)_{n=1}^\infty$ in $L^\infty(Y)$ such that $f_n \to f$ in $L^2(\nu_1)$, and $g_n \to g$ in $L^2(\nu_2)$.

By (4), the sequence of real numbers $(\int f_n \otimes g_n \, d\mu)_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows that, if we define

$$\int f \otimes g \, d\mu = \lim_{n\to\infty} \int f_n \otimes g_n \, d\mu, \tag{5}$$

then

$$\left| \int f \otimes g \, d\mu \right| \leqslant K_G \|\mu\|_{F_2} \|f\|_{L^2(\nu_1)} \|g\|_{L^2(\nu_2)}, \quad (f, g) \in L^2(\nu_1) \times L^2(\nu_2). \tag{6}$$

We now extend this idea. Let $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$ denote the completion of $L^2(\nu_1) \otimes L^2(\nu_2)$ in the projective tensor norm. Then for $\phi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$, we have

$$\|\phi\|_{\hat{\otimes}} = \inf\left\{ \sum_{j=1}^\infty \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)} \colon \phi = \sum_{j=1}^\infty f_j \otimes g_j \right\}, \tag{7}$$

where equality is according to the action on $L^2(\nu_1)^* \times L^2(\nu_2)^*$. (See Section IV.7 in [2], or Proposition 1.1.4 in [3].)

Equivalently, we can assume $\|f_j\|_{L^2(\nu_1)} \leqslant 1$, $\|g_j\|_{L^2(\nu_2)} \leqslant 1$, and there exist positive scalars λ_j such that $\sum_{j=1}^\infty \lambda_j < \infty$, and

$$\phi = \sum_{j=1}^\infty f_j \otimes g_j \lambda_j. \tag{8}$$

Proposition 4. *$L^\infty(X) \otimes L^\infty(Y)$ is dense in $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$ in the norm on $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$.*

Proof. Let $\epsilon > 0$ be given. Suppose that $\phi \in L^2(\nu_1) \otimes L^2(\nu_2)$. Then we can represent ϕ as in (8) with a finite sum; that is $\phi = \sum_{j=1}^N f_j \otimes g_j \lambda_j$, where $(f_j, g_j) \in L^2(\nu_1) \times L^2(\nu_2)$ such that $\|f_j\|_{L^2(\nu_1)} \leqslant 1$, $\|g_j\|_{L^2(\nu_2)} \leqslant 1$, for all $j \in \{1, \dots, N\}$, and $\lambda_j \geqslant 0$ are such that $\sum_{j=1}^N \lambda_j < \infty$. Let $\Lambda = \sum_{j=1}^N \lambda_j$.

There exist functions $(f_j')_{j=1}^N$ in $L^\infty(X)$ and $(g_j')_{j=1}^N$ in $L^\infty(Y)$ such that, for each $j \in \{1, \dots, N\}$,

$$\|f_j - f_j'\|_{L^2(\nu_1)} < \frac{\epsilon}{8\Lambda}, \qquad \|g_j - g_j'\|_{L^2(\nu_2)} < \frac{\epsilon}{8\Lambda}.$$

Without loss of generality, we may assume $\frac{\epsilon}{8\Lambda} < 1$, and thus $\|g_j'\|_{L^2(\nu_2)} < 2$ for all $j \in \{1, \dots, N\}$.

Let $\phi' = \sum_{j=1}^N f_j' \otimes g_j' \lambda_j \in L^\infty(X) \otimes L^\infty(Y)$. Then

$$\begin{aligned}
\left\| \sum_{j=1}^N f_j \otimes g_j \lambda_j - \sum_{j=1}^N f_j' \otimes g_j' \lambda_j \right\|_{\hat{\otimes}} &\leqslant \left\| \sum_{j=1}^N f_j \otimes g_j \lambda_j - \sum_{j=1}^N f_j \otimes g_j' \lambda_j \right\|_{\hat{\otimes}} + \left\| \sum_{j=1}^N f_j \otimes g_j' \lambda_j - \sum_{j=1}^N f_j' \otimes g_j' \lambda_j \right\|_{\hat{\otimes}} \\
&= \left\| \sum_{j=1}^N f_j \otimes (g_j - g_j') \lambda_j \right\|_{\hat{\otimes}} + \left\| \sum_{j=1}^N (f_j - f_j') \otimes g_j' \lambda_j \right\|_{\hat{\otimes}} \\
&\leqslant \sum_{j=1}^N \|f_j\|_{L^2(\nu_1)} \|g_j - g_j'\|_{L^2(\nu_2)} \lambda_j + \sum_{j=1}^N \|f_j - f_j'\|_{L^2(\nu_1)} \|g_j'\|_{L^2(\nu_2)} \lambda_j \\
&\leqslant \sum_{j=1}^N \frac{\epsilon}{8\Lambda} \lambda_j + \sum_{j=1}^N \frac{\epsilon}{8\Lambda} \cdot 2\lambda_j < \frac{\epsilon}{2}.
\end{aligned}$$

Now suppose $\psi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$. Then there exists a $\phi \in L^2(\nu_1) \otimes L^2(\nu_2)$ such that $\|\psi - \phi\|_{\hat{\otimes}} < \frac{\epsilon}{2}$. By the above argument, there exists a function $\phi' \in L^\infty(X) \otimes L^\infty(Y)$ such that $\|\phi - \phi'\|_{\hat{\otimes}} < \frac{\epsilon}{2}$. Therefore

$$\|\psi - \phi'\|_{\hat{\otimes}} \leqslant \|\psi - \phi\|_{\hat{\otimes}} + \|\phi - \phi'\|_{\hat{\otimes}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as required. $\square$

Definition 5. Suppose $\mu \in F_2(\mathcal{A}, \mathcal{B})$, and ν_1 and ν_2 are Grothendieck measures for μ. Let $\phi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$. Then

$$\int \phi \, d\mu = \lim_{n\to\infty} \int \phi_n \, d\mu, \tag{9}$$

where $(\phi_n)_{n=1}^{\infty}$ is a sequence in $L^\infty(X) \otimes L^\infty(Y)$ converging to ϕ in the norm on $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$.

The limit exists and is independent of defining sequence, because if $\phi \in L^\infty(X) \otimes L^\infty(Y)$, and $\phi = \sum_{j=1}^N f_j \otimes g_j$ is any representation with $(f_j)_{j=1}^N$ in $L^\infty(X)$ and $(g_j)_{j=1}^N$ in $L^\infty(Y)$, then by (4),

$$\left| \int \phi \, d\mu \right| \leqslant \sum_{j=1}^N \left| \int f_j \otimes g_j \, d\mu \right| \leqslant K_G \|\mu\|_{F_2} \sum_{j=1}^N \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)}. \tag{10}$$

The arguments closely parallel those of Proposition 4, and so will be omitted.

Proposition 6. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Let $\mu \in F_2(\mathcal{A}, \mathcal{B})$, and let ν_1 and ν_2 be associated Grothendieck probability measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. Then*

$$\left| \int \phi \, d\mu \right| \leqslant K_G \|\phi\|_{\hat{\otimes}} \|\mu\|_{F_2}, \quad \phi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2).$$

Proof. This follows from the definition in (9), and the inequality in (10). □

We now make some observations about $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$, where ν_1 and ν_2 are arbitrary probability measures on $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, respectively.

Lemma 7. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, and let ν_1 and ν_2 be probability measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\phi(x, y) = \sum_j f_j(x) g_j(y)$ a.e. $(\nu_1 \times \nu_2)$, and $\sum_j \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)} < \infty$, then $\phi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$.*

Proof. Let $\alpha \in L^2(\nu_1)$ and $\beta \in L^2(\nu_2)$. Then, by Fubini's theorem,

$$\begin{aligned}
\int_{X\times Y} \phi(x, y)\alpha(x)\beta(y)\,(\nu_1 \times \nu_2)(dx, dy) &= \int_{X\times Y} \left(\sum_j f_j(x) g_j(y) \right) \alpha(x)\beta(y)\,(\nu_1 \times \nu_2)(dx, dy) \\
&= \sum_j \left(\int_X f_j(x)\alpha(x)\,\nu_1(dx) \right) \left(\int_Y g_j(y)\beta(y)\,\nu_2(dy) \right) \\
&= \sum_j \langle f_j, \alpha \rangle_{L^2(\nu_1)} \langle g_j, \beta \rangle_{L^2(\nu_2)}.
\end{aligned}$$

The result follows. □

Theorem 8. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, and let H be a separable Hilbert space. Let ν_1 and ν_2 be probability measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. If $f : X \to H$ and $g : Y \to H$ are bounded, measurable functions, and*

$$\langle f, g \rangle(x, y) = \big\langle f(x), g(y) \big\rangle, \quad (x, y) \in X \times Y,$$

then $\langle f, g \rangle \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$.

Proof. Let $(e_j)_{j=1}^{\infty}$ be an orthonormal basis for H, and let $(f_j)_{j=1}^{\infty}$ and $(g_j)_{j=1}^{\infty}$ be the coordinate functions for f and g, respectively. Then

$$f(x) = \sum_{j=1}^{\infty} f_j(x) e_j, \quad \text{and} \quad g(y) = \sum_{j=1}^{\infty} g_j(y) e_j, \quad (x, y) \in X \times Y.$$

By assumption, $f_j \in L^\infty(X)$ and $g_j \in L^\infty(Y)$ for each $j \in \mathbb{N}$, and

$$\big\langle f(x), g(y) \big\rangle_H = \sum_{j=1}^{\infty} f_j(x) g_j(y), \quad (x, y) \in X \times Y.$$

We claim that $\sum_{j=1}^{\infty} \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)} < \infty$.

Let $N \in \mathbb{N}$ be arbitrary. By the choice of (f_j) and (g_j), for any $x \in X$ and $y \in Y$,

$$\sum_{j=1}^{N} \left|f_j(x)\right|^2 \leqslant \left\|f(x)\right\|_H^2, \quad \text{and} \quad \sum_{j=1}^{N} \left|g_j(y)\right|^2 \leqslant \left\|g(y)\right\|_H^2. \tag{11}$$

By the Cauchy–Schwarz inequality,

$$\sum_{j=1}^{N} \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)} \leqslant \left(\sum_{j=1}^{N} \|f_j\|_{L^2(\nu_1)}^2 \right)^{1/2} \left(\sum_{j=1}^{N} \|g_j\|_{L^2(\nu_2)}^2 \right)^{1/2}.$$

However, by (11),

$$\sum_{j=1}^{N} \|f_j\|_{L^2(\nu_1)}^2 = \sum_{j=1}^{N} \left(\int_X \left|f_j(x)\right|^2 \nu_1(dx) \right) \leqslant \int_X \left\|f(x)\right\|_H^2 \nu_1(dx),$$

and

$$\sum_{j=1}^{N} \|g_j\|_{L^2(\nu_2)}^2 = \sum_{j=1}^{N} \left(\int_Y \left|g_j(y)\right|^2 \nu_2(dy) \right) \leqslant \int_Y \left\|g(y)\right\|_H^2 \nu_2(dy).$$

Therefore, it follows that

$$\sum_{j=1}^{N} \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)} \tag{12}$$

$$\leqslant \left(\int_X \left\|f(x)\right\|_H^2 \nu_1(dx) \right)^{1/2} \left(\int_Y \left\|g(y)\right\|_H^2 \nu_2(dy) \right)^{1/2} \tag{13}$$

$$\leqslant \left(\sup_{x \in X} \left\|f(x)\right\|_H^2 \right)^{1/2} \left(\sup_{y \in Y} \left\|g(y)\right\|_H^2 \right)^{1/2}. \tag{14}$$

The functions f and g were assumed to be bounded, and so (12) is bounded uniformly in N.

It follows from Lemma 7 that $\langle f, g \rangle$ is in $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$, and furthermore, from (14),

$$\left\|\langle f, g \rangle\right\|_{\hat{\otimes}} \leqslant \sup_{x \in X} \left\|f(x)\right\|_H \sup_{y \in Y} \left\|g(y)\right\|_H. \tag{15}$$

This completes the proof. □

We are now prepared to prove the main result.

Theorem 9 *(The Grothendieck inequality). Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, and let H be a separable Hilbert space. Suppose $\mu \in F_2(\mathcal{A}, \mathcal{B})$. If $f : X \to H$ and $g : Y \to H$ are bounded (strongly) measurable functions, and*

$$\langle f, g \rangle(x, y) = \left\langle f(x), g(y) \right\rangle, \quad (x, y) \in X \times Y,$$

then

$$\int \langle f, g \rangle \, d\mu = \lim_{N \to \infty} \sum_{j=1}^{N} \left(\int f_j \otimes g_j \, d\mu \right),$$

where $(f_j)_{j \in \mathbb{N}}$ and $(g_j)_{j \in \mathbb{N}}$ are the coordinate functions for f and g, respectively. Furthermore,

$$\left| \int \langle f, g \rangle \, d\mu \right| \leqslant K_G \|\mu\|_{F_2} \sup_{x \in X} \left\|f(x)\right\|_H \sup_{y \in Y} \left\|g(y)\right\|_H.$$

Proof. We let ν_1 and ν_2 be the Grothendieck measures for μ. By Theorem 8, $\langle f, g \rangle$ is in $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$. The rest follows from Proposition 6 and (15). □

Our final proposition shows that the integral defined in (9) agrees with the standard definition for integration with respect to a bimeasure (as described in [2], for example). Let $V_2 = L^\infty(X) \hat{\otimes} L^\infty(Y)$ be the projective tensor product of $L^\infty(X)$ and $L^\infty(Y)$ equipped with the tensor norm

$$\|\phi\|_{V_2} = \inf\left\{ \sum_{j=1}^{\infty} \|f_j\|_{L^\infty(X)} \|g_j\|_{L^\infty(Y)} \colon \phi(x, y) = \sum_{j=1}^{\infty} f_j(x) g_j(y) \right\}, \tag{16}$$

where the infimum is taken over all pointwise representations of ϕ.

Proposition 10. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Let $\mu \in F_2(\mathcal{A}, \mathcal{B})$, and let ν_1 and ν_2 be associated Grothendieck probability measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose that $\phi \in V_2$. Then $\phi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$, and*

$$\|\phi\|_{\hat{\otimes}} \leqslant \|\phi\|_{V_2}.$$

Proof. Let $\epsilon > 0$ be arbitrary. There exists a representation of ϕ, say $\phi(x, y) = \sum_{j=1}^{\infty} f_j(x) g_j(y)$ for all $(x, y) \in X \times Y$, such that

$$\sum_{j=1}^{\infty} \|f_j\|_{L^\infty(X)} \|g_j\|_{L^\infty(Y)} \leqslant \|\phi\|_{V_2} + \epsilon.$$

The measures ν_1 and ν_2 are probability measures, and so

$$\sum_{j=1}^{\infty} \|f_j\|_{L^2(\nu_1)} \|g_j\|_{L^2(\nu_2)} \leqslant \sum_{j=1}^{\infty} \|f_j\|_{L^\infty(X)} \|g_j\|_{L^\infty(Y)}.$$

Therefore, for any $\epsilon > 0$,

$$\|\phi\|_{\hat{\otimes}} \leqslant \|\phi\|_{V_2} + \epsilon,$$

and hence the result. $\square$

4. Final remarks

Equality in $L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$ is determined by action on $L^2(\nu_1)^* \times L^2(\nu_2)^*$. The following shows that this equality is almost the same as pointwise equality.

Proposition 11. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, and let ν_1 and ν_2 be probability measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\phi \in L^2(\nu_1) \hat{\otimes} L^2(\nu_2)$. If ϕ has representations $\sum_j f_j \otimes g_j$ and $\sum_j \tilde{f}_j \otimes \tilde{g}_j$, then $\sum_j f_j(x) g_j(y) = \sum_j \tilde{f}_j(x) \tilde{g}_j(y)$ a.e. $(\nu_1 \times \nu_2)$.*

Proof. Let $E \in \mathcal{A}$ and $F \in \mathcal{B}$. The indicator functions $\mathbb{1}_E$ and $\mathbb{1}_F$ are in $L^2(\nu_1)$ and $L^2(\nu_2)$, respectively. Therefore, by assumption,

$$\sum_{j=1}^{\infty} \langle f_j, \mathbb{1}_E \rangle_{L^2(\nu_1)} \langle g_j, \mathbb{1}_F \rangle_{L^2(\nu_2)} = \sum_{j=1}^{\infty} \langle \tilde{f}_j, \mathbb{1}_E \rangle_{L^2(\nu_1)} \langle \tilde{g}_j, \mathbb{1}_F \rangle_{L^2(\nu_2)}.$$

Consequently,

$$\sum_{j=1}^{\infty} \int_{E \times F} f_j(x) g_j(y) \, (\nu_1 \times \nu_2)(dx, dy) = \sum_{j=1}^{\infty} \int_{E \times F} \tilde{f}_j(x) \tilde{g}_j(y) \, (\nu_1 \times \nu_2)(dx, dy).$$

By Fubini's theorem, this implies

$$\int_{E \times F} \left(\sum_{j=1}^{\infty} f_j \otimes g_j \right) d(\nu_1 \times \nu_2) = \int_{E \times F} \left(\sum_{j=1}^{\infty} \tilde{f}_j \otimes \tilde{g}_j \right) d(\nu_1 \times \nu_2).$$

Since E and F were arbitrary, the result follows. $\square$

It is worth noting that, despite Proposition 2, we have the following:

Proposition 12. *Let X and Y be compact Hausdorff spaces with Borel fields $\mathcal{A}$ and $\mathcal{B}$, respectively, and let H be a (separable) Hilbert space. If $f : X \to H$ and $g : Y \to H$ are continuous functions, then the function*

$$\langle f, g\rangle(x, y) = \big\langle f(x), g(y)\big\rangle, \quad (x, y) \in X \times Y,$$

is in $C(X) \hat{\otimes} C(Y)$.

Proof. Let $(f_j)_{j=1}^{\infty}$ and $(g_j)_{j=1}^{\infty}$ be the respective coordinate functions of f and g with respect to a given basis $(e_j)_{j=1}^{\infty}$ of H. Then, $\langle f(x), g(y)\rangle = \sum_{j=1}^{\infty} f_j(x)g_j(y)$, for all $(x, y) \in X \times Y$. It suffices to show the sequence $(\sum_{j=1}^{N} f_j \otimes g_j)_{N=1}^{\infty}$ is Cauchy in $C(X) \hat{\otimes} C(Y)$. To see this, let $\mu \in F_2(\mathcal{A}, \mathcal{B})$ be of norm 1. For any $N \geqslant M$, by the Grothendieck inequality,

$$\left| \int \left(\sum_{j=M}^{N} f_j \otimes g_j \right) d\mu \right| \leqslant K_G \sup_{x\in X} \left(\sum_{j=M}^{N} \big| f_j(x) \big|^2 \right)^{1/2} \sup_{y\in Y} \left(\sum_{j=M}^{N} \big| g_j(y) \big|^2 \right)^{1/2}. \tag{17}$$

By continuity and compactness, the right side of (17) is arbitrarily small when M and N are sufficiently large (independent of μ), and so the result follows from the duality $(C(X)\hat{\otimes}C(Y))^* = F_2(\mathcal{A}, \mathcal{B})$ (Theorem VI.13 in [2]). □

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